

# Saturated simple and 2-simple topological graphs with few edges

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## Abstract

A *simple topological graph* is a topological graph in which any two edges have at most one common point, which is either their common endpoint or a proper crossing. More generally, in a *k-simple topological graph*, every pair of edges has at most  $k$  common points of this kind. We construct *saturated* simple and 2-simple graphs with few edges. These are  $k$ -simple graphs in which no further edge can be added. We improve the previous upper bounds of Kynčl, Pach, Radoičić, and Tóth [4] and show that there are saturated simple graphs on  $n$  vertices with only  $7n$  edges and saturated 2-simple graphs on  $n$  vertices with  $14.5n$  edges. As a consequence,  $14.5n$  edges is also a new upper bound for  $k$ -simple graphs (considering all values of  $k$ ). We also construct saturated simple and 2-simple graphs that have some vertices with low degree.

## 1 Introduction

Let  $G = (V, E)$  be a finite simple graph. A *drawing* of  $G$  is a map  $\delta: V \cup E \rightarrow \mathbb{R}^2$  that is one-to-one on  $\delta|_V: V \rightarrow \mathbb{R}^2$ , i.e.,  $\delta$  assigns the vertices of the graph to different points of the plane. Furthermore, we require that  $\delta|_E: E \rightarrow \mathcal{C}$ , where  $\mathcal{C}$  is a set of “nice” non-self-intersecting curves with two boundary points of the plane. For example we might think of  $\mathcal{C}$  as the set of all Jordan curves or, more elementary, of the set of all simple polygonal curves. For simplicity, we will not distinguish between an edge and the curve on which it is embedded, and between a vertex and the point on which it is embedded. We assume

that for any  $e = xy \in E$  the edge  $\delta(e)$  is a curve connecting  $\delta(x)$  and  $\delta(y)$  and it doesn't go through any other vertex, and also that any two different edges meet at finitely many points and any meeting point — that is not a common endvertex — is a proper crossing of the two curves.

The pair  $(G, \delta)$ , i.e., a graph with a drawing, is called a *topological graph*. A topological graph  $(G, \delta)$  is *simple* if in  $\delta$  two edges have at most one common point. More generally, the topological graph is called *k-simple* if in  $\delta$  two edges have at most  $k$  common points. For both simple and *k-simple* graphs we do not allow self-intersecting edges. A topological graph is a *geometric graph* if all its edges are drawn as straight-line segments. Obviously, every geometric graph is simple, provided that the vertices are placed in general position. Thus, every graph has simple drawings.

For a graph property  $\mathcal{T}$ , a graph  $G$  is  $\mathcal{T}$ -saturated if  $G$  has property  $\mathcal{T}$ , but the addition of any edge joining two non-adjacent vertices of  $G$  violates property  $\mathcal{T}$ . Often structures with property  $\mathcal{T}$  are quite hard to grasp, but  $\mathcal{T}$ -saturated structures might have a more useful character. We direct the interested reader to applications of the saturation technique [1, 3, 5]. This notion can be naturally extended to hypergraphs. A thorough survey by Faudree, Faudree, and Schmitt [2] discusses the case when property  $\mathcal{T}$  is “not having  $F$  as a sub(hyper)graph”.

In this paper we study *saturated k-simple* topological graphs. These are topological graphs that are *k-simple*, but no edge can be added without violating the *k-simplicity* of the drawing. Saturated planar drawings are triangulations and have therefore due to Euler's formula  $3n - 6$  edges. Recently, Kynčl, Pach, Radoičić, and G. Tóth [4] started to investigate saturated simple *k-simple* graphs. The maximum number of edges a saturated simple topological graph can have is clearly  $\binom{n}{2}$ , since the geometric graph of  $K_n$  with vertices in general position is a simple drawing. The more intriguing questions ask about the minimum number of edges for saturated *k-simple* topological graph. One of the main results of Kynčl et al. [4] is a construction of sparse saturated simple and *k-simple* topological graphs. We denote by  $s_k(n)$  the minimum number of edges a saturated *k-simple* graph with  $n$  vertices can have. Their upper bound on  $s_k(n)$  is a linear function of  $n$ , for  $n$  being the number of vertices; see Table 1 for the bounds obtained by Kynčl et al. [4]. The gap between the best known upper and lower bounds for  $s_k(n)$  is quite substantial. We only know that  $s_1(n) \geq 1.5n$  and that  $s_k(n) \geq n$  [4].

**Our contribution.** We improve the upper bounds for  $s_k(n)$  for  $k = 1, 2$ . We do this by showing that for any positive integer  $n$  there exists a saturated simple topological graph with at most  $7n$  edges (in Sect. 2), and a saturated 2-simple graph with at most  $14.5n$  edges (in Sect. 3). Sections 2 and 3 are independent. This result also implies that there are saturated *k-simple* graphs with at most  $14.5n$  edges for every  $k$ . See also Table 1 for a comparison with the old bounds. Our proofs are constructive, i.e., we can explicitly present the sparse saturated graphs.

We complete our results by studying *local saturation* of topological graphs. Here,

$k$	1	2	3	4	5	6,8,10	7	9, $\geq 11$
old upper bounds [4]	$17.5n$	$16n$	$14.5n$	$13.5n$	$13n$	$9.5n$	$10n$	$7n$
new upper bounds	<b><math>7n</math></b>	<b><math>14.5n</math></b>						

Table 1: Old and new upper bounds for  $s_k(n)$ , the minimum number of edges in a saturated *k-simple* graph with  $n$  vertices.

local saturation refers to drawings in which one (or several) vertices have a small vertex degree even though the full drawing might not be the sparsest. Such observations might be helpful in further studies, e.g., if we want to estimate techniques for proving lower bounds that are based on the minimum vertex degree in saturated graphs. We show that there are saturated simple graphs that have a vertex of degree 4, and saturated simple graphs in which 10 percent of the vertices have degree 5. For saturated 2-simple graphs we can prove that there are drawings with minimum degree 12. The current lower bounds for  $s_k(n)$  are obtained by bounding the minimum vertex degree in saturated  $k$ -simple graphs [4]. Our results show the limits of this approach.

## 2 Saturated simple topological graph with few edges

In this section we give a construction that generates sparse saturated simple graphs. We start with defining a graph  $G$ , parametrized by an integer  $k$ , with  $n = 6k$  vertices and  $9k - 6$  edges. This graph is the backbone of our sparse saturated graph.

The drawing is best visualized on the surface of a long circular cylinder. Fig. 1 shows an unrolling of the cylinder into the plane. The cylinder is obtained by cutting the drawing along the two dotted lines and gluing the top and the bottom together. The vertices of the graph are placed in a  $3 \times 2k$ -grid-like fashion. We draw the vertices together in pairs, with each vertex  $X_i^L$  on the *left* and the corresponding vertex  $X_i^R$  on the *right*, for  $X = A, B, C$  and  $i = 1, \dots, k$ . We refer to the vertices whose label have the subscript  $i$  as the  $i$ -th layer.  $G$  is the union of

- three vertex-disjoint paths of *blue edges* connecting  $A_1^L A_2^L \dots A_k^L$ ,  $B_1^L B_2^L \dots B_k^L$ , and  $C_1^L C_2^L \dots C_k^L$ ,
- three vertex-disjoint paths of *red edges* connecting  $A_1^R A_2^R \dots A_k^R$ ,  $B_1^R B_2^R \dots B_k^R$ , and  $C_1^R C_2^R \dots C_k^R$ , and
- $k$  disjoint cycles of *green edges* connecting  $A_i^L B_i^L C_i^L$ .

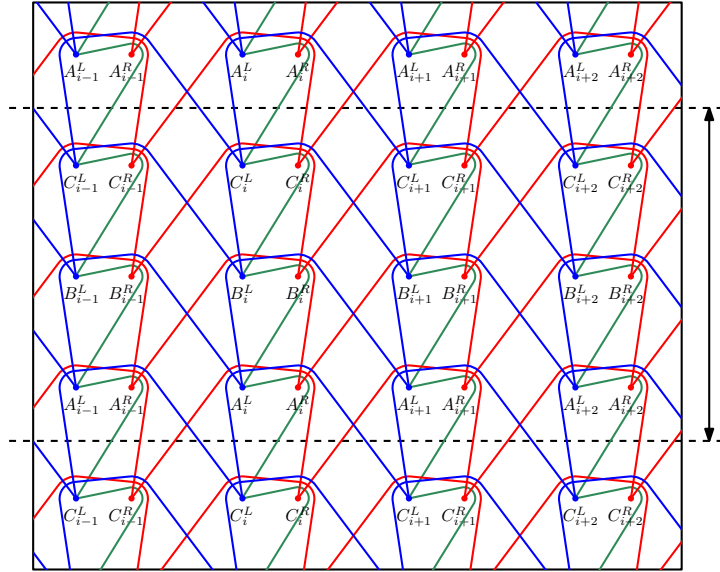


Figure 1: The graph  $G$  on an unrolled cylinder.

The cylinder can be homeomorphically mapped into the plane, as shown in Fig. 2 for the red and blue edges only. The horizontal directions turn into radial directions. But the resulting drawings suffer from large distortions, and the left-right symmetry is lost. We therefore prefer the cylindrical drawings, and we extend the cylinder surface periodically beyond the dotted lines (using the plane as a universal cover of the cylinder). One should however be aware that vertices (and edges) that appear as distinct in the figure may denote the same vertex, as indicated by the vertex labels.

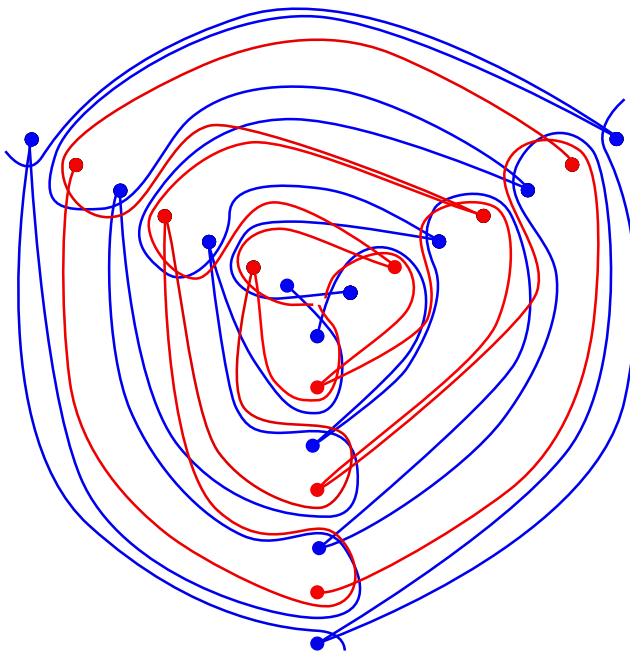


Figure 2: The graph  $G$  on the plane.

We will first consider the graph  $G_{RB}$  that omits the green edges, because this graph is more symmetric: with the exception of the vertices  $X_1^{L/R}$  and  $X_k^{L/R}$  near the boundary, all vertices look identical. Apart from these boundary effects, the drawing has a rotational symmetry, cyclically shifting the labels  $A \rightarrow B \rightarrow C \rightarrow A$ , a translational symmetry, shifting indices  $i$  up or down, and a mirror symmetry, exchanging left with right and blue with red. The green edges destroy this mirror symmetry: there are then two classes of vertices, the blue vertices  $X_i^L$  and the red vertices  $X_i^R$ .

Let  $G_{RB}$  denote the topological graph obtained by restricting  $G$  to the red and blue edges. We will show that the maximum degree in any saturated drawing which extends  $G_{RB}$  is 16. The 16 potential neighbors of a typical vertex  $A_i^L$  are shown in Fig. 3. This establishes that there are saturated drawings with  $n$  vertices and less than  $8n$  edges. When the green edges are included, the three dashed edges in Fig. 3 become impossible. Thus, each blue vertex has 13 potential neighbors. The red vertex  $A_{i+1}^R$ , which can be taken as a representative of a typical red vertex, loses  $A_i^L$  as a potential neighbor. Thus, each red vertex has at most 15 potential neighbors. This improves the upper bound for the smallest number of edges in a saturated drawings with  $n$  vertices to  $7n$ .

**Theorem 1.** *Let  $s(n)$  denote the minimum number of edges that a simple saturated drawing with  $n$  vertices can have. Then*

$$s(n) \leq 7n.$$

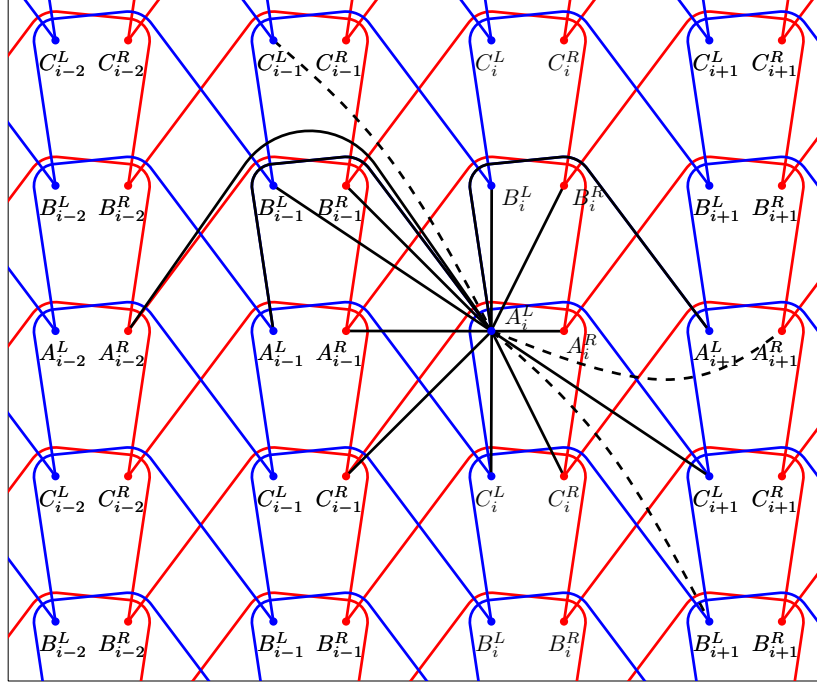


Figure 3: The 16 potential neighbors of a vertex.

The remainder of this section is devoted to proving the above theorem. We start with the analysis of the graph  $G_{RB}$ .

## 2.1 The graph $G_{RB}$

**Lemma 1.** *The 16 potential neighbors of a typical vertex  $A_i^L$  in  $G_{RB}$  are all 11 vertices of levels  $i-1$  and  $i$  ( $A_{i-1}^L, B_{i-1}^L, C_{i-1}^L; A_{i-1}^R, B_{i-1}^R, C_{i-1}^R; B_i^L, C_i^L; A_i^R, B_i^R, C_i^R$ ) plus the 5 vertices  $A_{i-2}^R; A_{i+1}^L, B_{i+1}^L, C_{i+1}^L; A_{i+1}^R$ .*

When any of the neighbors listed above does not exist because  $i \leq 2$  or  $i = k$ , the lemma still holds in the sense that the remaining vertices form the set of potential neighbors. In the proofs, when we exclude an edge between, say, levels  $i$  and  $j$ , our arguments will not use edges outside this range.

In the following we will look at the given drawing of  $G_{RB}$  (or  $G$ ) and argue about the additional edges that can be drawn. The implicit assumption is that these edges cannot cross any given edge more than once. Usually, we will regard a new edge as a directed edge, starting at some vertex and trying to reach another vertex.

A *belt* is a substructure of our drawing. It is formed by the 12 vertices of two successive layers with their 6 edges between them, see Fig. 4. This drawing separates a large face on the left from a large face on the right. More precisely, the belt is defined as the part of the plane (or the cylinder) which lies between these two large faces (shaded area).

We denote the six edges of the belt by  $\alpha^L, \beta^L, \gamma^L, \alpha^R, \beta^R, \gamma^R$ ; as shown in Fig. 4. Each edge is cut into six sections by the intersections with the other edges: Two sections are little “stumps” at the end vertices. One section belongs to the boundary between the belt and the *outside*. The remaining three sections form the *top part* of the edge. We say that a new (directed) edge crosses a belt edge *from the outside* or *from the top* if it crosses the boundary part or the top part in the appropriate direction.

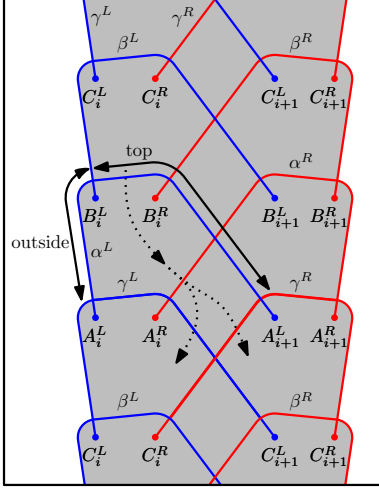


Figure 4: Escape from a belt is difficult (Lemma 2).

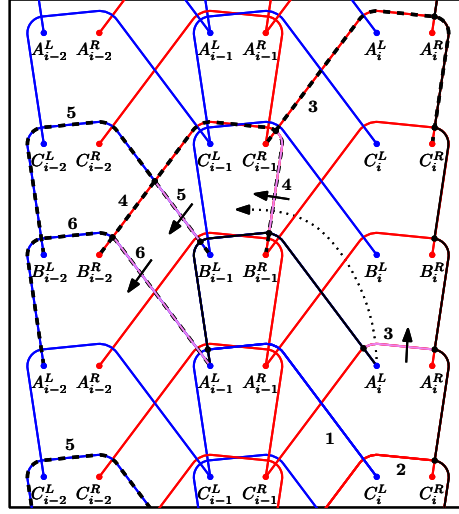


Figure 5: The situation discussed in the proof of Lemma 1 for left side neighbors.

**Lemma 2.** *In a simple drawing that contains  $G_{RB}$ , the following holds:*

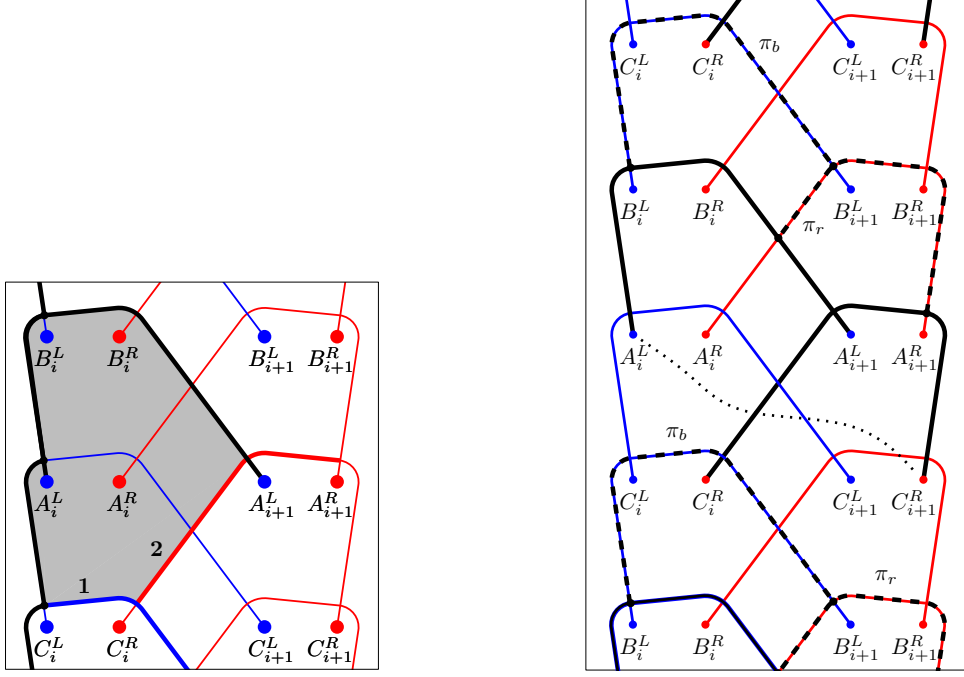
1. *If an edge crosses a belt edge from the top or from the outside, it must terminate inside the belt.*
2. *No edge can cross a belt.*

*Proof.* We start with the following observation: If an edge crosses  $\alpha_L$  from the outside or from the top, and it does not terminate at  $B_i^L$  or at  $B_i^R$ , then it must later cross  $\gamma^L$  or  $\gamma^R$  from the top. This observation holds symmetrically for  $\alpha_R$  instead of  $\alpha_L$ , and cyclically for the other four belt edges. Hence, any edge that “enters” the belt from the outside has to continue by crossing another edge from the belt from the top. There is no way to leave the belt without crossing some edge twice.  $\square$

After these preparations, we are ready to prove Lemma 1.

*Proof of Lemma 1.* Let us first look at the potential neighbors on the left side. A connection from  $A_i^L$  to levels  $j \leq i - 3$  is impossible, because it would have to cross a belt. For the vertices at level  $i - 2$  we observe the following (see Fig. 5 for the edge numbers we are referring to): When we start from  $A_i^L$  we cannot cross the right boundary of the belt formed by levels  $i - 1$  and  $i$ , because then we would have to cross the whole belt to reach level  $i - 2$ . If we cross edge 1 or 2 from the top, then, by Lemma 2, we are restricted to the belt defined by level  $i - 1$  and  $i$ . Thus we can regard edge 1 and 2 as closed from the top. (These edges can later be crossed from the bottom.) We successively conclude that the new edges must cross the purple parts of the edges 3, 4, 5, and 6. The endpoints  $B_{i-2}^R, B_{i-2}^L, A_{i-2}^L$  of the edges 4, 5, and 6 cannot be taken.  $C_{i-2}^L$  and  $C_{i-2}^R$  are enclosed in a small face delimited by the edges 4, 5, and 6, and cannot be reached.  $A_{i-2}^R$  is thus the only reachable vertex of level  $i - 2$ .

Let us turn to the potential neighbors on the right side. A connection from  $A_i^L$  to levels  $j \geq i + 3$  is impossible, because it would have to cross a belt. Vertices at level  $i + 2$  cannot be reached either, because (i) if we cross the edge forming the left boundary of the belt spanned by the vertices of level  $i$  and  $i + 1$  we cannot cross this belt anymore and therefore cannot reach level  $i + 2$ , and (ii) if we cross one of the edge in the face that



(a) Level  $i + 2$  cannot be reached from  $A_i^L$ .

(b)  $C_{i+1}^R$  cannot be reached from  $A_i^L$ .

Figure 6: Restricting the neighbors to the right.

contains  $A_i^L$  from the top (edge labeled 1 and 2 in Fig. 6a), then, by Lemma 2, we are also restricted to this belt. Thus we are restricted to the shaded region in Fig. 6a.

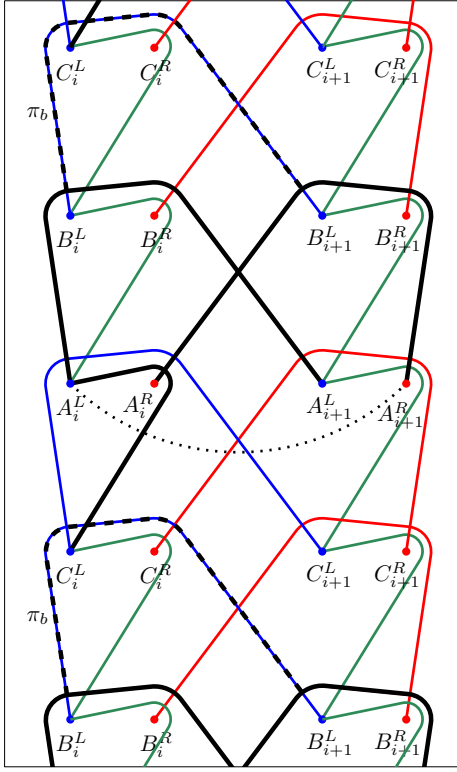
The vertices  $B_{i+1}^R$  and  $C_{i+1}^R$  also cannot be neighbors of  $A_i^L$ . We discuss the exclusion of  $C_{i+1}^R$  as a potential neighbor – the case for  $B_{i+1}^R$  is symmetric. The edges incident to  $A_i^L$  and  $C_{i+1}^R$ , which we call the *closed edges* cannot be crossed. The closed edges are depicted as thicker curves in Fig. 6b. Consider the portion of the red edge  $\pi_r$  that runs between  $A_i^R$  and  $A_{i+1}^R$  above the closed edges (see Fig. 6b). The curve  $\pi_r$  bounds a region below in which the remaining edges bounding this region are parts of the closed edges. Hence, if we enter this region we cannot leave and therefore we cannot cross  $\pi_r$  (see Fig. 6b). Let us now consider the partial edge  $\pi_b$  that runs between  $B_{i+1}^L$  and  $B_i^L$  above the closed edges and  $\pi_r$ . Again, there is a region whose boundary is part of the closed edges and also  $\pi_b$ . To enter and leave this region we have to cross either one of the closed edges or  $\pi_r$ , or we have to cross  $\pi_b$  twice. Since all these options are invalid, we have to avoid this region, and therefore are not allowed to cross  $\pi_b$ . We observe that the closed edges together with  $\pi_b$  and  $\pi_r$  leave  $A_i^L$  and  $C_{i+1}^R$  in different faces, which shows that these vertices cannot be neighbors unless we cross one edge twice.  $\square$

## 2.2 The Graph $G$

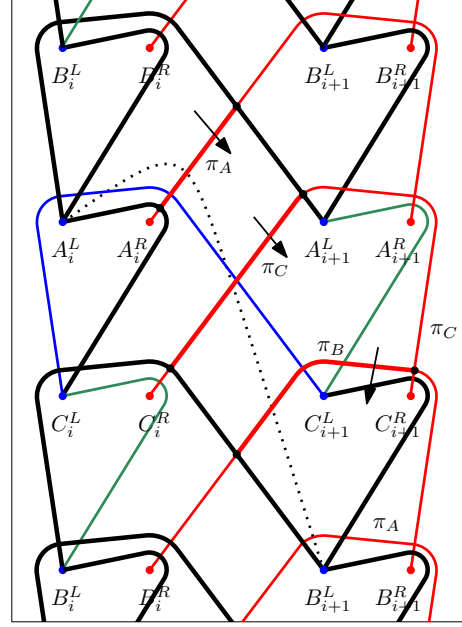
Now we turn back to  $G$ . The additional green edges exclude some of the possible edges from the Lemma 1. To analyze the drawing of  $G$  we have to treat the left and right vertices differently.

**Lemma 3.** 1. The 13 potential neighbors of a typical vertex  $A_i^L$  in  $G$  are all 5 vertices of level  $i$  ( $B_i^L, C_i^L; A_i^R, B_i^R, C_i^R$ ), all but one vertex of level  $i - 1$  ( $A_{i-1}^L, B_{i-1}^L; A_{i-1}^R, B_{i-1}^R, C_{i-1}^R$ ) plus the 3 vertices  $A_{i-2}^R; A_{i+1}^L, C_{i+1}^L$ .





(a)  $A_i^L$  and  $A_{i+1}^R$  cannot be connected.



(b)  $A_i^L$  and  $B_{i+1}^L$  cannot be connected.

Figure 7: Restricting more neighbors by putting back the green edges.

2. The 15 potential neighbors of a typical vertex  $A_i^R$  in  $G$  are all 11 vertices of levels  $i$  and  $i+1$  ( $A_i^L, B_i^L, C_i^L; B_i^R, C_i^R; A_{i+1}^L, B_{i+1}^L, C_{i+1}^L; A_{i+1}^R, B_{i+1}^R, C_{i+1}^R$ ) plus the 4 vertices  $A_{i-1}^R, B_{i-1}^R, C_{i-1}^R; A_{i+2}^L$ .

The claim immediately follows from the next two lemmas.

**Lemma 4.** In a simple extension of  $G$ ,  $A_{i+1}^R$  cannot be a neighbor of  $A_i^L$ .

*Proof.* We call the edges incident to  $A_{i+1}^R$  and  $A_i^L$  the *closed edges*. Let  $\pi_b$  the portion of the edge connecting  $B_i^L$  with  $B_{i+1}^L$  that runs above the closed edges (see Fig. 7a). The cell “below”  $\pi_b$  is only bounded by  $\pi_b$  and the closed edges. Hence, we cannot leave this cell once we have entered. As a consequence we cannot cross  $\pi_b$ . Since the closed edges together with  $\pi_b$  disconnect  $A_{i+1}^R$  and  $A_i^L$ , these two vertices cannot be neighbors.  $\square$

**Lemma 5.** In a simple extension of  $G$ ,  $B_{i+1}^L$  cannot be a neighbor of  $A_i^L$ .

*Proof.* All edges that are incident to either  $B_{i+1}^L$  or  $A_i^L$  cannot be crossed. These edges are drawn as black curves in Fig. 7b and are now considered as being the *closed edges*. The only chance to connect  $A_i^L$  with  $B_{i+1}^L$  is to enter the region that is bounded by the closed edges and the edge  $\pi_A$  from  $A_i^R$  to  $A_{i+1}^R$ . Thus we have to cross this edge to leave this face. This leads us to a region that is bounded by the closed edges,  $\pi_A$  and the edge  $\pi_C$  from  $C_i^R$  to  $C_{i+1}^R$ . Clearly we have to cross  $\pi_C$  to leave this region. Now we have entered a region that is bounded by a closed edge,  $\pi_C$  and the edge  $\pi_B$  that connects  $B_i^R$  with  $B_{i+1}^R$ . To leave this region we have to cross  $\pi_B$ , which brings us to a region that is bounded by a closed edge,  $\pi_A$  and  $\pi_C$ . We observe that we are stuck in this region and hence, cannot reach  $B_{i+1}^L$ .  $\square$



By symmetry,  $C_{i-1}^L$  and  $A_i^L$  cannot be neighbors, and this concludes the proof of Lemma 3. Moreover, as a consequence of Lemma 3 the average degree in a saturated extension of  $G$  is at most 14, which proves Theorem 1 when the number  $n$  of vertices is a multiple of 6.

We can determine the vertex degrees more carefully. If  $k \geq 3$ , then

1. the degrees of  $A_1^L, B_1^L, C_1^L$  are at most 7,
2. the degrees of  $A_1^R, B_1^R, C_1^R$  are at most 12,
3. the degrees of  $A_2^L, B_2^L, C_2^L$  are at most 12,
4. the degrees of  $A_i^R, B_i^R, C_i^R$  are at most 15, when  $1 < i < k-1$ ,
5. the degrees of  $A_i^L, B_i^L, C_i^L$  are at most 13, when  $2 < i < k$ ,
6. the degrees of  $A_{k-1}^R, B_{k-1}^R, C_{k-1}^R$  are at most 14,
7. the degrees of  $A_k^L, B_k^L, C_k^L$  are at most 11,
8. the degrees of  $A_k^R, B_k^R, C_k^R$  are at most 8.

A straightforward calculation gives that any saturated extension of  $G$  has at most  $7n - 30$  edges. For  $k = 2$ , the degrees of  $X_1^L, X_1^R, X_2^L, X_2^R$  are bounded by 7, 11, 10, 8, respectively, for a total of 54 edges, which also agrees with the formula  $7n - 30$ . Hence, for any  $n \geq 12$  that is a multiple of 6, there exists a saturated simple topological graph with  $n$  vertices and at most  $7n - 30$  edges.

Our construction can be extended to any vertex size by *cloning* some vertices. Take a saturated simple topological graph and any vertex  $P$  of it. Next to  $P$  we add  $\rho$  new copies of  $P$  – the clones. Connect the neighbors of  $P$  to each clone by edges that are non-intersecting perturbations of the edges incident to  $P$ . By this we obtain a simple drawing. A saturation of this drawing can include as additional edges only edges among  $P$  and its clones.

For  $n \geq 12$ , we can write  $n$  as  $6k + \rho$  where  $0 \leq \rho \leq 5$ . If  $\rho = 0$ , we are done. If  $\rho \geq 1$ , then start with a construction for a saturated simple topological graph with  $6k$  vertices. Add  $\rho$  clones of its lowest-degree vertex  $P$ , and saturate. In our construction, the lowest degree is 7. Cloning such a vertex  $\rho$  times adds up to  $7\rho + \binom{\rho+1}{2}$  additional edges after saturation. Since  $\rho \leq 5$ , the number of edges is bounded by

$$7(6k) - 30 + 7\rho + \binom{\rho+1}{2} \leq 7(6k + \rho) - 30 + 15 = 7(6k + \rho) - 15 < 7n$$

The resulting simple topological graph proves Theorem 1 for  $n \geq 12$ . If  $n \leq 11$ , then the bound of Theorem 1 holds since even the complete graph has at most  $\binom{n}{2} \leq 5n$  edges.

### 3 Saturated 2-simple topological graphs with few edges

In this section we construct sparse saturated drawings in the 2-simple case. We first review an auxiliary structure called grid-block. Then we use it to construct an efficient blocking edge configuration. We finish with explicit constructions of sparse saturated 2-simple drawings.

### 3.1 The grid-block configuration

To begin, we study a drawing of 6 edges (three red edges and three black edges) as depicted in Fig. 8. The drawing consists of three disjoint horizontal segments representing the red edges  $r_1, r_2, r_3$ , and three disjoint black edges  $b_1, b_2, b_3$  that are drawn such that one crosses (in order)  $r_1, r_2, r_3, r_1, r_2, r_3$ , the other  $r_2, r_3, r_1, r_2, r_3, r_1$ , and the last one  $r_3, r_1, r_2, r_3, r_1, r_2$ . There are no other crossings in the drawing. Note that the configuration superimposes a grid. We call such an arrangement of edges a *grid-block*. These blocks have been used by Kynčl et al. as building blocks in their saturated graphs [4]. In the terminology of Kynčl et al., our grid-blocks are named (3,2)-grid-blocks.

As done in the previous section we consider the graph as drawn on the cylinder. More precisely, we draw the graph inside a rectangle in which we identify two sides in opposition (*bottom side* and *top side*), while the other sides are named *right side* and *left side*. If an edge uses the transition across the bottom/top edge we say that it *wraps around*. In the following we assume that the grid-blocks are drawn such that only the black edges wrap around. We label every face of the drawing of a grid-block with 2 numbers. These numbers refer to the coordinates of the (dual) superimposed grid, with  $(0,0)$  being the label of the face that contains the two bottom most endpoints of the black edges on the left side. All “vertical” coordinates are considered modulo 3.

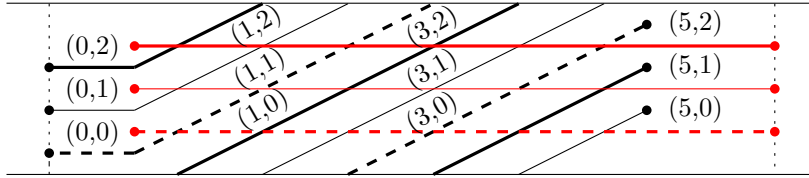


Figure 8: A grid-block with some labeled faces.

Throughout the section we study *paths* connecting the left and the right sides of the cylinder and passing through some blocking configurations. By a *path* in this context we always mean a path in a graph dual to the arrangement of the blocking configuration in question.

Kynčl et al. observed that every path connecting the left with the right side of the cylinder has to intersect the edges of grid-block at least 5 times. For our construction we need a stronger statement which is presented in Lemma 6. The following lemma simplifies the treatment of paths passing through the grid-block.

**Lemma 6.** *Let  $\gamma$  be a path crossing the grid-block that starts in face  $(0,i)$  and ends in face  $(5,j)$  and that never visits the faces  $(0,\cdot)$ ,  $(5,\cdot)$  again, see Fig. 9. Then  $\gamma$  can be transformed, keeping its endpoints fixed, to a path  $\tilde{\gamma}$  such that  $\tilde{\gamma}$ :*

1. *crosses (with the same or smaller multiplicity) only the edges of the grid-block crossed by  $\gamma$ ,*
2. *first walks between the faces  $(0,i)$ ,  $0 \leq i \leq 2$ , then crosses some black edges to the right, passing from a face  $(k,i)$  to a face  $(k+1,i)$ , then crosses some red edges upwards, passing from a face  $(k,i)$  to a face  $(k+1,i+1)$ .*

*Proof.* We refer to the transition of the path from one cell of the arrangement to an adjacent cell as a *step*. There are four different types of steps:  $\rightarrow$ ,  $\leftarrow$ ,  $\nearrow$  or  $\swarrow$ , depending on the crossed edge and the direction, see Fig. 9.

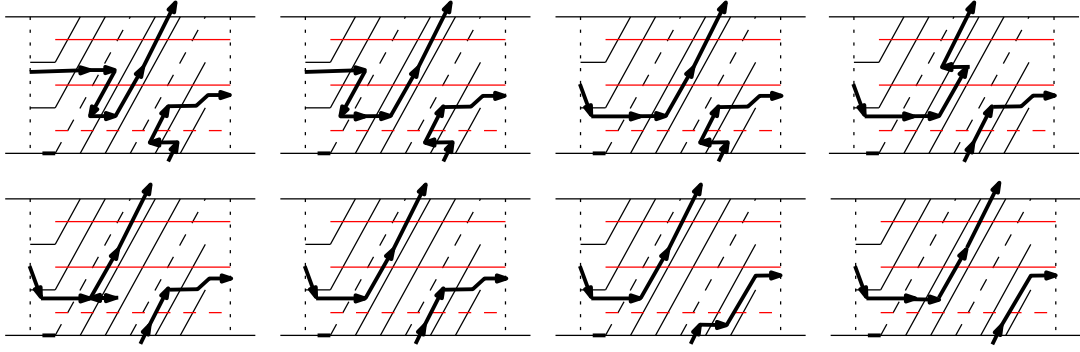


Figure 9: Process of the simplification of a path passing through a grid-block.

We execute the path simplification through a series of local modifications on pairs of two consecutive steps: (1) annihilation of two consecutive steps in opposite directions and (2) changing places of two consecutive steps that are not yet in a desired order.

The simplification is carried out in two stages. In the first stage (shown in the first 6 pictures in Fig. 9) we remove all “backward steps”  $\swarrow$  and  $\leftarrow$ , while possibly increasing the number of steps the path  $\tilde{\gamma}$  walks between faces  $(0, i)$ ,  $0 \leq i \leq 2$ . In the second stage we reorder the steps  $\nearrow$  and  $\rightarrow$  such that no  $\nearrow$  precedes any  $\rightarrow$ .

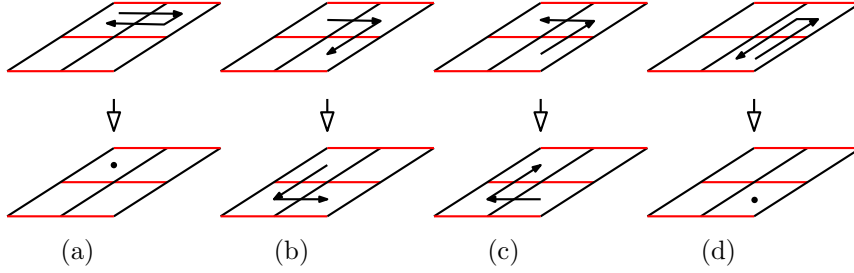


Figure 10: 4 possible 2-step configurations involving “backward steps” as a second step before (first row) and after (second row) the appropriate local modification.

**Stage 1:** We traverse the path until we meet the first  $\leftarrow$  or  $\swarrow$  step. Together with its preceding step it forms one of the 4 configurations shown in Fig. 10. In cases (a) and (d) the steps only differ in their orientation, hence we can annihilate two steps. In the remaining cases (b) and (c) we reorder the two steps. This reordering can be safely executed unless it forces the path to leave the grid-block. This, however, may happen only when the backward step ( $\leftarrow$  or  $\swarrow$ ) starts from one of the faces labeled  $(2, k)$ . Since this backward step is the first backward step of the path, we are left with two subcases for each (b) and (c) depending on the preceding step, which might be either  $\rightarrow$  or  $\nearrow$ . The four cases are depicted in Fig. 11. All the cases can be handled by further local simplifications that are shown in the figure.

We finish the proof of the stage 1 using double induction on the number of backward steps and, within it, on the distance from the beginning of the path to the first backward step.

**Stage 2:** After the stage 1 our path through the grid, leaving aside its first steps between faces  $(0, i)$ , has only  $\rightarrow$  and  $\nearrow$  steps. These two types of steps can be reordered without changing the number of times the path crosses any edge of the grid. Moreover this reordering never leads the path out of the grid-block.  $\square$

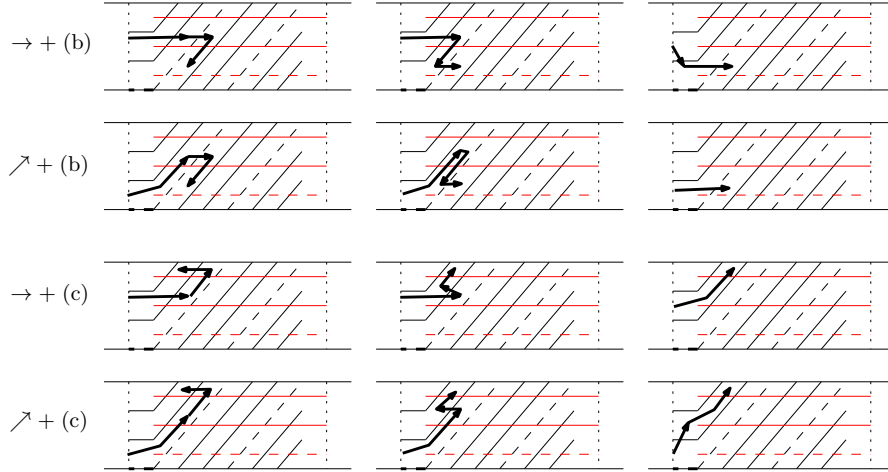


Figure 11: Handling of the 4 possible cases (each row represents one case) when the local modifications (b) or (c) force the path out of the grid-block.

### 3.2 A blocking configuration

We call the building blocks of the following constructions *black block* and *red block*, see Fig. 12. We refer to the edges of the red (black) block as *red edges* (*black edges*). Any two red edges, as well as any two black edges, cross exactly twice. Note that up to a reflection the red block is homeomorphic to the black block.

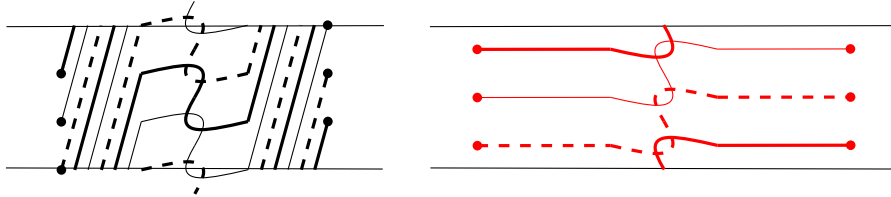


Figure 12: A black (left) and a red (right) blocks.

We combine two black blocks and a red block as shown in Fig. 13 to obtain a drawing that we call a *3-block*. Since the red block differs from the black block only by a reflection, the 3-block built from consecutive black-red-black blocks is a mirror image of the 3-block built from consecutive red-black-red blocks.

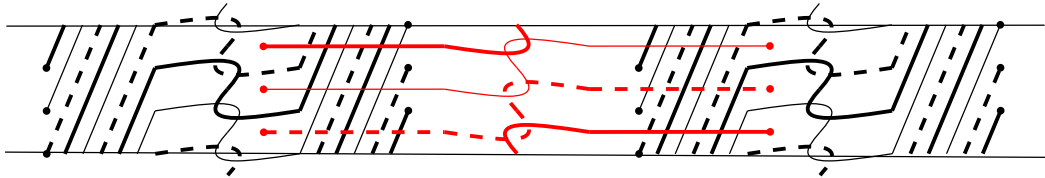


Figure 13: A 3-block, formed by consecutive black, red and again black blocks.

The following theorem is the key observation that we need for the construction of the sparse 2-simple drawing.

**Theorem 2.** *Any path connecting the left with the right sides of the cylinder while passing through the 3-block crosses one of the edges forming the 3-block at least 3 times.*

Before proving the theorem we provide some helpful lemmas. We label some of the faces of the arrangement as shown in Fig. 14. In particular, for  $i = 0, 1, 2$ , we denote the faces containing the left endpoint of the red edges  $r_i$  as  $L_i$ , and the faces containing the right endpoint as  $R_i$ . The edges of the left black block are named  $b_i$  and the edges of the right black block are named  $b'_i$ . Finally, let  $LM_i$  be the face that contains the right endpoint of  $b_i$ , and let  $RM_i$  be the face that contains the left endpoint of  $b'_i$ . The region spanned by  $L_0$ ,  $L_1$  and  $L_2$  is denoted by  $L$ . We similarly define regions  $LM$ ,  $RM$  and  $R$ .

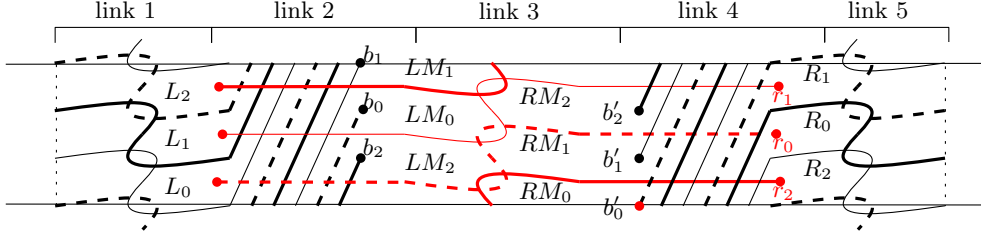


Figure 14: A 3-block with some distinguished faces (capital letters) and edges. The red edges forming the blocks are labeled  $b_i$ ,  $b'_i$  and  $r_i$ . The “zones” at which we subdivide the path into links are labeled above the strip.

Let  $\gamma$  be a path that passes the 3-block. To facilitate the analysis we subdivide the path  $\gamma$  into smaller pieces, which we call *links*. The links are defined as follows:

- link 1: from the start point (left) of  $\gamma$  to the last point of  $\gamma$  in  $L$ ,
- link 2: from the last point of  $\gamma$  in  $L$  to its first point in  $LM$ ,
- link 3: from the first point of  $\gamma$  in  $LM$  to its last point in  $RM$ ,
- link 4: from the last point of  $\gamma$  in  $RM$  to its first point in  $R$ ,
- link 5: from the first point of  $\gamma$  in  $R$  to its (right) endpoint.

Before we proceed we check that the links are well defined, i.e., that the points defining the links appear in order. For the links 1, 3 and 5 this holds trivially, while to check it for links 2 (and, symmetric, 4), we need to prove that the last point in  $L$  precedes the first point in  $LM$ :

**Lemma 7.** *No path can visit the regions  $L \rightarrow LM \rightarrow L \rightarrow LM$  in this order without crossing some of the edges forming the 3-block at least 3 times.*

*Proof.* The faces  $L$  and  $LM$  are separated by a grid-block. Passing through it requires at least 5 crossings of its edges. Any path visiting  $L \rightarrow LM \rightarrow L \rightarrow LM$  would cross the grid-block at least 3 times, and hence it would cross the edges of the grid-block at least  $3 \times 5 = 15$  times. Since a grid-block is formed by 6 edges, at least one of them will be crossed 3 times or more.  $\square$

We continue by analyzing the path through the 3-block following its links.

**Lemma 8.** *Any path passing the 3-block from left to right with the last point of link 1 at  $L_i$  crosses the edge  $b_{i+1}$  at least once or one of the edges  $b_i$  and  $b_{i+2}$  at least twice at its first link (all indices modulo 3).*

*Proof.* A path that ends in  $L_i$  crosses either  $b_{i+1}$  or it crosses  $b_{i+2}$  while entering from  $L_{i+1}$ . Repeating this argument twice proves the lemma.  $\square$

The following lemma summarizes the behavior of the path on the first two links:

**Lemma 9.** *Any path  $\gamma$  passing the 3-block that does not intersect any edge 3 times or more crosses the red edges  $r_j, r_{j+1}$  before it first visits the region  $LM$  at  $LM_j$ .*

*Proof.* We modify the path  $\gamma$  along link 2 following the simplification procedure described in Lemma 6 to get a path  $\tilde{\gamma}$ . Lemma 6 also implies that the link 2 of  $\tilde{\gamma}$  consists of exactly 5 “steps”: first,  $0 \leq h \leq 5$  steps crossing the black edges  $\rightarrow$  to the right, followed by  $v = 5 - h$  steps crossing red edges  $\nearrow$  upward.

Assume that the first point of link 2 of  $\tilde{\gamma}$  lies inside the face  $L_i$ . Then  $h$  horizontal steps of link 2 cross the  $b_{i+1}, b_i, b_{i-1}, \dots, b_{i+1-(h-1)}$ . Moreover, Lemma 8 guarantees that already link 1 of the path  $\tilde{\gamma}$  crossed either  $b_{i+1}$  once or one of  $b_i$  or  $b_{i+2}$  twice. Since  $\tilde{\gamma}$  does not cross any of the black edges more than twice, it follows that  $h \leq 3$ . This, however, shows that  $v \geq 2$ , which implies that the path  $\tilde{\gamma}$  crosses the red edges  $r_{j+1}, r_j$  before it reaches the last point of its second link in face  $LM_j$ . To finish the proof we recall that the path  $\gamma$  crosses every edge of the 3-block at least as many times as  $\tilde{\gamma}$  and that the last points of the link 2 of  $\gamma$  and  $\tilde{\gamma}$  coincide.  $\square$

*Proof of Theorem 2.* We prove by contradiction, namely, we assume that there is a path  $\gamma$  that passes through the 3-block while crossing every edge of the 3-block at most twice. Let  $LM_j$  be the face where link 2 ends, and let  $RM_k$  be the face where link 4 starts. By Lemma 9 we know that  $\gamma$  crosses  $r_j$  and  $r_{j+1}$  in link 1 and link 2. Since the structure of the link 4 and 5 coincides with the structure of link 2 and 1 we can apply Lemma 9 also to the last 2 links. Thus,  $\gamma$  crosses  $r_{k-1}, r_k$  in link 4 and 5. A short case distinction ( $k$  might be either  $j, j+1$ , or  $j+2$ ) shows that  $\gamma$  cannot connect endpoints of link 2 and 4 via link 3 without crossing at least one of the red edges 3 times; see Fig. 15. The figure depicts all ways of how to possibly route the path  $\gamma$  in link 3. Each of the possible continuations crosses some of the red edges  $r_j, r_{j+1}, r_{j-1}$  twice and is blocked within one of the faces before it reaches the face  $RM_k$ . As a consequence the path  $\gamma$  cannot exist.  $\square$

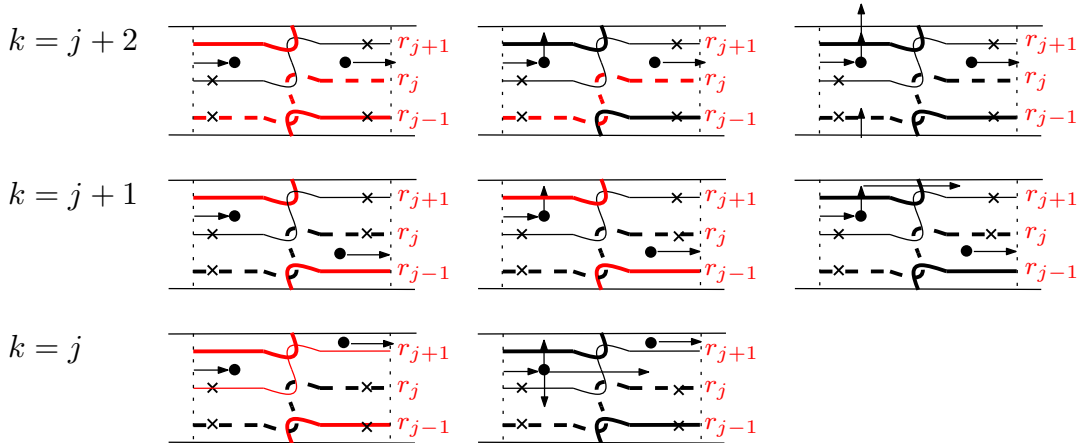


Figure 15: Each row depicts a case. Black dots inside faces mark the faces  $LM_j$  (left) and  $RM_k$  (right). Black crosses on red edges mark the edges that are, due to Lemma 9, crossed by the path outside link 3. We color red edges black as soon as they are crossed by the path  $\gamma$  twice and no more crossings are allowed. In the case  $k = j$  the path can be continued in 3 different direction, in each of them the path is blocked after one step.

### 3.3 A sparse saturated 2-simple drawing

We show next how to combine a sequence of 3-blocks in order to obtain a 2-simple saturated drawing with few edges to obtain the following result.

**Theorem 3.** *Let  $s_2(n)$  denote the minimum number of edges that a 2-simple saturated drawing with  $n$  vertices can have. Then  $s_2(n) \leq 14.5n$ .*

*Proof.* We consider the drawing that repeats the pattern shown in Fig. 16. The horizontal strip denotes the cylinder. The drawing is formed by  $k$  consecutive black and red blocks; see Fig. 12. Each block contains 6 vertices, so the total number of vertices is  $k \times 6$ . Clearly, the drawing is 2-simple.

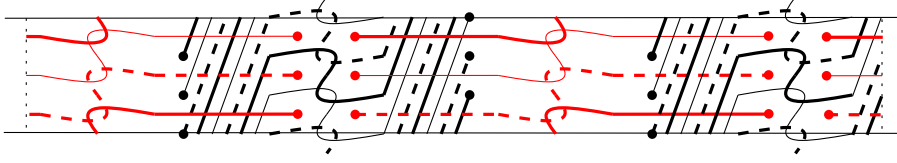


Figure 16: A 2-simple drawing that does not allow too many edges to be added.

Now we add as many edges as possible without violating the 2-simplicity, so that the drawing becomes saturated (this padding procedure is definitely not unique). Theorem 2 implies that without violating the 2-simplicity any vertex can be connected by an edge only to 29 other vertices; see Fig. 17 for “internal” vertices and Fig. 18 for vertices close to the left (right) boundary of the cylinder. This implies that the maximal number of edges in the resulting saturated 2-simple drawing is less or equal than  $14.5n$ .

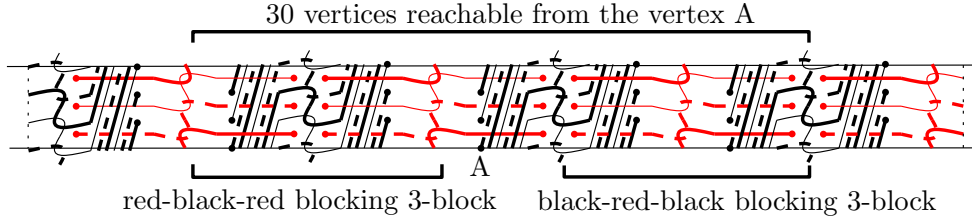


Figure 17: The potential neighbors of a typical vertex A.

For  $n$  not divisible by 6 we build the construction above with  $k = \lfloor n/6 \rfloor$ . We split the remaining  $l = n - 6\lfloor n/6 \rfloor$  vertices into two groups of no more than 3 vertices each, and place one group with  $l_1$  vertices to the left and one group with  $l_2$  vertices on to the right of the resulting arrangement.

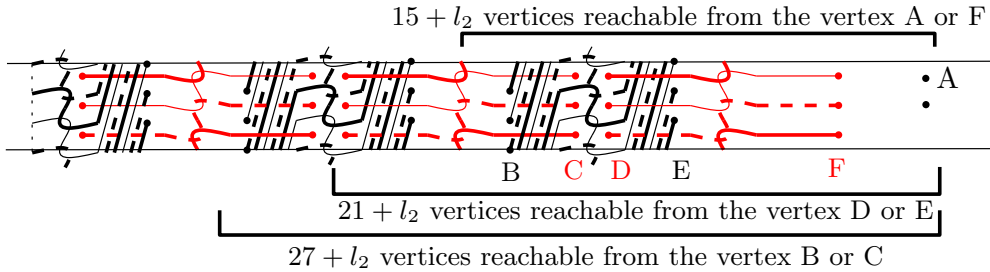


Figure 18: The potential neighbors of vertices close to the boundary.

The possible connections with the newly introduced vertices are illustrated in Fig. 18. Since  $l_1, l_2 \leq 3$ , no vertex has degree greater than 29.  $\square$



## 4 Local saturation

### 4.1 Simple drawings

The lower bound in [4] on the number of edges in a saturated simple topological graph is based on the following lemma.

**Lemma 10** ([4]). *Let  $G$  be a simple topological graph with at least four vertices, and let  $A$  be a vertex of degree at most two. Then  $G$  has a simple extension by an edge incident to  $A$ .*

This lemma implies that in a simple saturated topological graph with at least four vertices, every vertex must have degree at least three, and hence the number of edges is at least  $1.5n$ . Can we improve the bound on the edge number by strengthening the lower bound on the degree? The following considerations establish a limit to this approach: There are saturated graphs with minimum degree four.

We say that a vertex  $S$  in a simple topological graph is *saturated* if it cannot be connected to a non-adjacent vertex while maintaining simplicity. The above lemma implies that in a simple topological graph with at least four vertices, a saturated vertex must have degree at least three.

**Observation 1.** *For any positive integer  $n \geq 6$ , there is a simple topological graph on  $n$  vertices with a saturated vertex of degree four.*

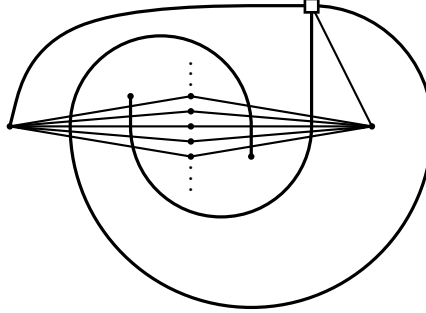


Figure 19: The boxy vertex of degree four is saturated.

The observation is due to the construction presented in Fig. 19. This example is an extension of the case  $n = 6$  from [4, Fig. 2]. The topmost vertex is saturated since all incident faces are bounded by edges incident to that vertex.

The following lemma presents a construction that realizes small vertex degrees for many vertices.

**Lemma 11.** *For any positive integer  $k$ , there exists a saturated simple topological graph on  $10k$  vertices with  $k$  vertices of degree 5.*

*Proof.* The main idea of our construction is depicted in Fig. 20. A simple case distinction verifies that no edge can connect the central vertex with a point on the outer face without violating the simplicity of drawing.

Now, take  $k$  copies of the drawing in Fig. 20, and place them on the plane next to each other such that the interior faces of the copies are non-overlapping. The  $k$  copies of the central vertex will remain degree-5 vertices no matter how we saturate the graph.  $\square$

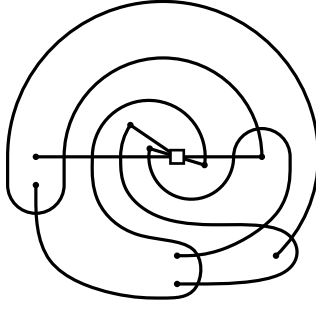


Figure 20: In the simple topological graph above, the central vertex has degree 5, and it cannot be connected by an edge to any point in the unbounded region while keeping simplicity.

## 4.2 2-simple drawings

To study local saturation in 2-simple case we use a slight modification of the 3-block introduced in Sect. 3; see Fig. 21.

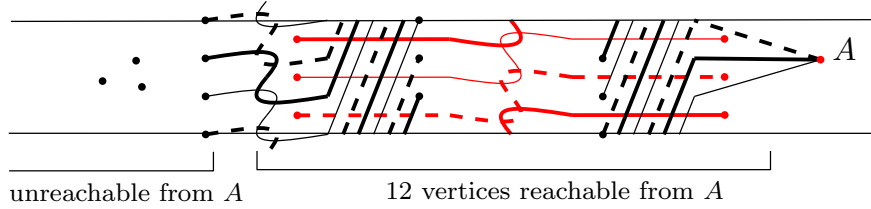


Figure 21: The rightmost vertex  $A$  cannot be connected to any vertex that belongs to the leftmost (unbounded) face without violating 2-simplicity.

By the arguments given in the proof of Theorem 2 the rightmost vertex can be connected to only 12 other vertices (Fig. 21) and thus cannot be connected to any vertex that belongs to the leftmost (unbounded) face of the drawing without violating 2-simplicity.

The “unrolling” of this configuration from the cylinder to the plane (with center of the unrolling in the rightmost vertex) is presented on Fig. 22. The central vertex cannot be connected by an edge to any vertex that belongs to the unbounded region without violating 2-simplicity, and so it has degree no larger than 12 in any saturation. After placing  $k$  disjoint copies of this construction to the plane next to each other we obtain the following result:

**Lemma 12.** *For any positive integer  $k$ , there exists a saturated 2-simple topological graph on  $16k$  vertices with  $k$  vertices of degree 12.*

## Acknowledgment

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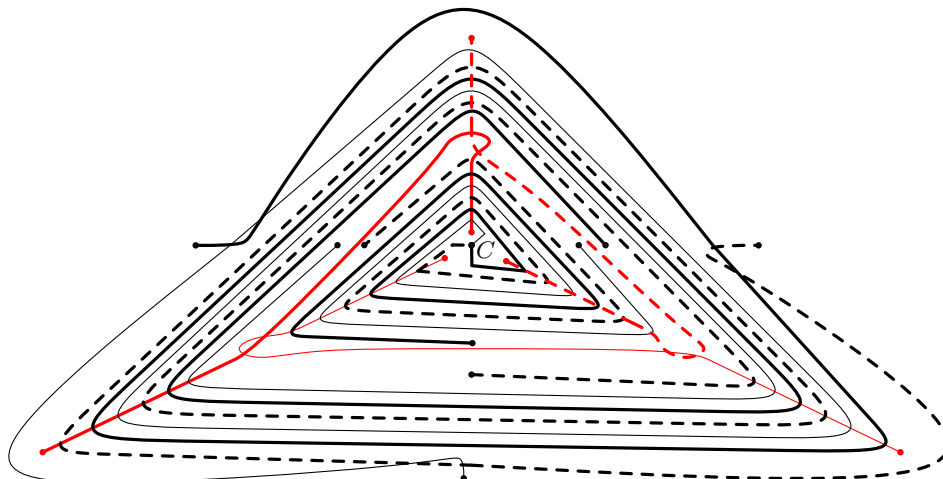


Figure 22: Unrolling of Fig. 21 to the plane. The central vertex  $C$  corresponds to the rightmost vertex  $A$  of Fig. 21.

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